# ON THE MOTION OF THE SATELLITE INERTIA CENTER IN THE EARTH'S GRAVITATIONAL FIELD

## (O DVIZHENII TSENTRA INERTSII SPUTNIKA V POLE TIAGOTENIIA ZEMLI)

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First works on the motion of an Earth's satellite utilized the classical perturbation theory of the elliptical elements of the orbit. The gravitational force potential contained the first nontrivial term of the potential expansion in spherical functions [1 and 2]. The additional terms were taken into account in [3 and 4]. A new qualitative effect was discovered which does not appear in the first approximation. Another direction is connected with the approximate representation of the Earth's potential by an expression which permits the separation of variables in the Jacobi-Hamilton equations [5 to 8]. A great deal of attention is devoted to the problem of satellite motion in [9].

The present paper derives new equations of motion, the form of which is suitable for application of averaging and the small parameter methods. Utilization of spherical coordinates leads to lowering of the equations' order. Study of the orbital elements is relegated to second place. Considered are the simplest forms of motion. Use of paper [10] is made in deriving the equations of motion (*).

**1. Introductory notation.** We introduce a fixed system of coordinates $Oxyz$ with origin at the Earth's center and unit vectors $\mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3$. The $z$-axis is the axis of Earth's rotation and is directed towards the North Pole. The spherical system of coordinates $r, \vartheta, \lambda$ is introduced

$$x = r \sin \vartheta \cos \lambda, \qquad y = r \sin \vartheta \sin \lambda, \qquad z = r \cos \vartheta \qquad (1.1)$$

The coordinate set of the spherical system of coordinates is given by the unit vectors $\mathbf{e}_r, \mathbf{e}_\vartheta, \mathbf{e}_\lambda$. The Earth's mass distribution is assumed such that the gravitationsl field potential does not contain $\lambda$ and is an even function of $\gamma$ $(\gamma = \cos \vartheta)$. The latter assumption is not essential for the greatest part of the paper.

*) A.I. Lur'e. Certain nonlinear problems of the dynamic of space flight. Presentation. Nonlinear Problems of Space Flight. Third Conference on Nonlinear Oscillations. Berlin, 1964.

The material point $M$ of unit mass (satellite) moves only due to the action of the Earth's gravitational forces; the influence of the atmosphere, the Moon, the Sun, etc. is neglected.

Let $\mathbf{r}$ be the radius vector of point $M$. We introduce the orbital set $\mathbf{e}_r$, $\mathbf{e}_\varphi$, $\mathbf{n}$. Vector $\mathbf{e}_r$ is parallel and codirectional with the vector $\mathbf{r}$, $\mathbf{r} = \mathbf{e}_r r$. Vector $\mathbf{n}$ is perpendicular to the vectors $\mathbf{r}$, $d\mathbf{r}/dt$ and is codirectional with $\mathbf{r} \times d\mathbf{r}/dt$. Vector $\mathbf{e}_\varphi$ is defined by the equality

$$\mathbf{e}_\varphi = \mathbf{n} \times \mathbf{e}_r$$

**2. The initial equations of motion.** The equations of motion for the point $M$ are of the form

$$d^2\mathbf{r} / dt^2 = -\operatorname{grad} \Pi \tag{2.1}$$

Let us introduce into consideration the angular momentum

$$\mathbf{k} = \mathbf{r} \times d\mathbf{r} / dt, \qquad |\mathbf{k}| = k, \qquad \mathbf{k} = \mathbf{n}k \tag{2.2}$$

By dot multiplication by $\mathbf{e}_r$ Equation (2.1) is transformed into

$$\frac{d^2r}{dt^2} - \frac{k^2}{r^3} = -\frac{\partial \Pi}{\partial r} \tag{2.3}$$

It follows from (2.1) and (2.2) that

$$d\mathbf{k} / dt = -\mathbf{r} \times \operatorname{grad} \Pi \tag{2.4}$$

Substituting $\mathbf{r} = r\mathbf{e}_r$ into (2.2) and multiplying vectorially by $\mathbf{e}_r$, the equation for $\mathbf{e}_r$ is obtained

$$d\mathbf{e}_r / dt = r^{-2}k\mathbf{n} \times \mathbf{e}_r \tag{2.5}$$

Substituting $\mathbf{k} = k\mathbf{n}$, $\mathbf{r} = r\mathbf{e}_r$ into (2.4) and dot multiplying by $\mathbf{n}$ the following equation is obtained

$$dk / dt = -r\mathbf{e}_\varphi \cdot \operatorname{grad} \Pi \qquad (\mathbf{e}_\varphi = \mathbf{n} \times \mathbf{e}_r) \tag{2.6}$$

Multiplying Equation (2.4) twice vectorially by $\mathbf{n}$ there results

$$\frac{d\mathbf{n}}{dt} = -\frac{r}{k}(\mathbf{n} \cdot \operatorname{grad} \Pi)\, \mathbf{e}_r \times \mathbf{n} \tag{2.7}$$

Introduce the vector

$$\boldsymbol{\omega}_1 = kr^{-2}\mathbf{n} - rk^{-1}(\mathbf{n} \cdot \operatorname{grad} \Pi)\, \mathbf{e}_r \tag{2.8}$$

Equations (2.5),(2.7) can be rewritten into the form

$$\frac{d\mathbf{e}_r}{dt} = \boldsymbol{\omega}_1 \times \mathbf{e}_r, \qquad \frac{d\mathbf{n}}{dt} = \boldsymbol{\omega}_1 \times \mathbf{n} \tag{2.9}$$

i.e. the vector $\boldsymbol{\omega}_1$ is the angular velocity vector of the orbital set $\mathbf{e}_r$, $\mathbf{e}_\varphi$, $\mathbf{n}$.

Equations (2.9), (2.3) and (2.5) form a complete system of equations of motion of ninth order [10] with the three known relationships

$$\mathbf{e}_r \cdot \mathbf{e}_r = 1, \qquad \mathbf{e}_r \cdot \mathbf{n} = 0, \qquad \mathbf{n} \cdot \mathbf{n} = 1 \tag{2.10}$$

**3. First form of the equations of motion.** We introduce the angular velocity vector $\boldsymbol{\omega}_2$ of the set $\mathbf{e}_r, \mathbf{e}_\vartheta, \mathbf{e}_\lambda$ of the spherical system of coordinates

$$\boldsymbol{\omega}_2 = \mathbf{i}_3\lambda^{\cdot} + \mathbf{e}_\lambda\vartheta^{\cdot} \qquad (\lambda^{\cdot} \equiv d\lambda/dt) \tag{3.1}$$

We will denote by $\chi$ an angle formed by the instantaneous orbital plane containing $\mathbf{r}$ and $\dot{\mathbf{r}}$, and the meridianal plane containing $\mathbf{i}_3$ and $\mathbf{r}$. The sets $\mathbf{e}_r, \mathbf{e}_\varphi, \mathbf{n}$ and $\mathbf{e}_r, \mathbf{e}_\vartheta, \mathbf{e}_\lambda$ have in common the vector $\mathbf{e}_r$. Their angular velocities $\omega_1$ and $\omega_2$ can differ only by the terms $\mathbf{e}_r\chi^{\cdot}$, i.e.

$$\frac{k}{r^2}\mathbf{n} - \frac{r}{k}(\mathbf{n}\cdot\mathrm{grad}\,\Pi)\,\mathbf{e}_r = \lambda^{\cdot}\mathbf{i}_3 + \vartheta^{\cdot}\mathbf{e}_\lambda + \chi^{\cdot}\mathbf{e}_r \tag{3.2}$$

Projecting the equality (3.2) on $\mathbf{e}_r, \mathbf{e}_\vartheta, \mathbf{e}_\lambda$, there results the scalar equalities

$$-\frac{r}{k}\mathbf{n}\ \mathrm{grad}\,\Pi = \lambda^{\cdot}\cos\vartheta + \chi^{\cdot}$$

$$-\frac{k}{r^2}\sin\chi = -\lambda^{\cdot}\sin\vartheta, \qquad \frac{k}{r^2}\cos\chi = \vartheta^{\cdot} \tag{3.3}$$

We introduce a new independent variable $\tau$ and a dependent variable $u$ in accordance with

$$d\tau = kr^{-2}dt = ku^2dt, \qquad u = r^{-1} \tag{3.4}$$

In the problem of Keplerian motion, the angle of true anomaly corresponds to the nonholonomic variable $\tau$. Differentiation with respect to $\tau$ will be denoted by a prime. From (3.3) we have

$$\chi' = \Omega - \lambda'\cos\vartheta, \qquad \lambda'\sin\vartheta = \sin\chi, \qquad \vartheta' = \cos\chi \tag{3.5}$$

Here is introduced the notation

$$\Omega = -\frac{r^3}{k^2}\mathbf{n}\cdot\mathrm{grad}\,\Pi = -\frac{r^3}{k^2}\mathbf{n}\cdot\left(\frac{\partial\Pi}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial\Pi}{\partial\vartheta}\mathbf{e}_\varphi\right) =$$

$$= -\frac{r^2}{k^2}\frac{\partial\Pi}{\partial\vartheta}\mathbf{n}\cdot\mathbf{e}_\varphi = \frac{1}{k^2u^2}\frac{\partial\Pi}{\partial\vartheta}\sin\chi \tag{3.6}$$

Let us introduce a new variable

$$\gamma = \cos\vartheta \tag{3.7}$$

Differentiating $\gamma$ with respect to $\tau$, we find on the strength of (3.5) and (3.6)

$$\gamma' = -\sin\vartheta\cos\chi, \qquad 1 - \gamma^2 - \gamma'^2 = \sin^2\vartheta\sin^2\chi$$

$$\gamma'' = -\cos\vartheta + \sin\vartheta\sin\chi\Omega = -\gamma + \sin^2\vartheta\sin^2\chi\frac{1}{k^2u^2\sin\vartheta}\frac{\partial\Pi}{\partial\vartheta} \tag{3.8}$$

The differential equation for $\gamma$ follows from (3.8)

$$\gamma'' + \gamma = -(1 - \gamma^2 - \gamma'^2)\frac{1}{hu^2}\frac{\partial\Pi}{\partial\gamma} \qquad (h = k^2) \tag{3.9}$$

In Equation (2.3), we substitute (3.4) and obtain

$$u'' + u = -\frac{1}{h}\frac{\partial\Pi}{\partial u} - \frac{1}{2h}u'h' \tag{3.10}$$

From (2.6) and (3.4) we find

$$h' = \frac{dk^2}{dt}\frac{r^2}{k} = -2r^3\mathbf{e}_\varphi\cdot\mathrm{grad}\,\Pi = -2r^2\frac{\partial\Pi}{\partial\vartheta}\cos\chi \tag{3.11}$$

From (3.7) and (3.8) we finally obtain

$$h' = -\frac{2r^2}{\sin\vartheta}\frac{\partial\Pi}{\partial\vartheta}\sin\vartheta\cos\chi = -\frac{2}{u^2}\frac{\partial\Pi}{\partial\gamma}\gamma' \tag{3.12}$$

Equations (3.12) and (3.9) have an integral which expresses the constancy of the projection of angular momentum on the $z$-axis.

$$h\,(1-\gamma^2-\gamma'^2)=\sigma \qquad (\sigma=\text{const}) \tag{3.13}$$

Eliminating $h$ from (3.9) and (3.10) with the aid of (3.12) and (3.13) we get

$$u''+u=-\frac{1-\gamma^2-\gamma'^2}{\sigma}\left(\frac{\partial\Pi}{\partial u}-\frac{1}{u^2}\frac{\partial\Pi}{\partial\gamma}u'\gamma'\right)$$

$$\gamma''+\gamma=-\frac{1}{\sigma u^2}(1-\gamma^2-\gamma'^2)^2\frac{\partial\Pi}{\partial\gamma} \tag{3.14}$$

These equations admit the energy integral

$$\sigma\,(u^2+u'^2)=2\,(E-\Pi)\,(1-\gamma^2-\gamma'^2) \qquad (E=\text{const}) \tag{3.15}$$

Eliminating the independent variable $\tau$ (this can be done with the aid of the principle of least action in the Jacobi form ([11] p.712), there results one equation of second order. This, apparently, does not facilitate the investigation.

The present section derived in greater detail Equations (3.14) as suggested by Lur'e in the Presentation.noted previously.

4. **Second form of the equations of motion.** Equation (2.6) can be transformed into

$$\frac{d\mathbf{k}}{dt}=-r\mathbf{e}_r\times\left(\frac{\partial\Pi}{\partial r}\mathbf{e}_r+\frac{1}{r}\frac{\partial\Pi}{\partial\vartheta}\mathbf{e}_\vartheta\right)=\mathbf{e}_\lambda\sin\vartheta\,\frac{\partial\Pi}{\partial\gamma} \tag{4.1}$$

In Equations (4.1), (2.5) and (2.3), we pass to the new independent variable $\varphi$ and dependent variable $u$

$$d\varphi=r^{-2}dt=u^2dt, \qquad u=r^{-1} \tag{4.2}$$

We obtain Equations

$$\frac{d\mathbf{k}}{d\varphi}=\frac{\sin\vartheta}{u^2}\frac{\partial\Pi}{\partial\gamma}\mathbf{e}_\lambda, \qquad \frac{d\mathbf{e}_r}{d\varphi}=\mathbf{k}\times\mathbf{e}_r \tag{4.3}$$

$$\frac{d^2u}{d\varphi^2}+hu=-\frac{\partial\Pi}{\partial u}, \qquad h=k^2 \tag{4.4}$$

Let us express the vectors $\mathbf{k}$, $\mathbf{e}_r$ by the projections on the fixed axes of the rectangular coordinates $x$, $y$, $z$

$$\mathbf{k}=k_1\mathbf{i}_1+k_2\mathbf{i}_2+k_3\mathbf{i}_3, \qquad \mathbf{e}_r=\gamma_1\mathbf{i}_1+\gamma_2\mathbf{i}_2+\gamma\mathbf{i}_3$$
$$\gamma_1=\sin\vartheta\cos\lambda, \qquad \gamma_2=\sin\vartheta\sin\lambda, \qquad \gamma=\cos\vartheta \tag{4.5}$$

Substituting (4.5) into (4.3), we get the scalar equations

$$\frac{dk_1}{d\varphi}=-\frac{\gamma_2}{u^2}\frac{\partial\Pi}{\partial\gamma}, \qquad \frac{dk_2}{d\varphi}=\frac{\gamma_1}{u^2}\frac{\partial\Pi}{\partial\gamma}, \qquad \frac{dk_3}{d\varphi}=0 \tag{4.6}$$

$$\frac{d\gamma_1}{d\varphi}=k_2\gamma-k_3\gamma_2, \qquad \frac{d\gamma_2}{d\varphi}=k_3\gamma_1-k_1\gamma, \qquad \frac{d\gamma}{d\varphi}=k_1\gamma_2-k_2\gamma_1 \tag{4.7}$$

The eighth order system consisting of Equations (4.4), (4.6) and (4.7) describes the motion of the point $M$. There are known two obvious relationships

$$\gamma_1^2 + \gamma_2^2 + \gamma_3^2 = 1, \qquad \gamma_1 k_1 + \gamma_2 k_2 + \gamma k_3 = 0 \tag{4.8}$$

Differentiating (4.7) with respect to $\varphi$ we find on the strength of (4.6) to (4.8)

$$\frac{d^2\gamma_1}{d\varphi^2} + \left(h - \frac{\gamma}{u^2}\frac{\partial\Pi}{\partial\gamma}\right)\gamma_1 = 0, \qquad \frac{d^2\gamma_2}{d\varphi^2} + \left(h - \frac{\gamma}{u^2}\frac{\partial\Pi}{\partial\gamma}\right)\gamma_2 = 0 \tag{4.9}$$

$$\frac{d^2\gamma}{d\varphi^2} + h\gamma = (\gamma^2 - 1)\, u^{-2}\frac{\partial\Pi}{\partial\gamma}, \qquad h = k_1^2 + k_2^2 + k_3^2 \tag{4.10}$$

Differentiating $h$ with respect to $\varphi$, we get on the strength of (4.6) and (4.7)

$$\frac{dh}{d\varphi} = -\frac{2}{u^2}\frac{\partial\Pi}{\partial\gamma}\frac{d\gamma}{d\varphi} \tag{4.11}$$

Equations (4.4), (4.10) and (4.11) represent a complete system of fifth order; the order can be reduced to second utilizing the "autonomy" and two known integrals corresponding to (3.13) and (3.15)

$$h(1 - \gamma^2) - \left(\frac{d\gamma}{d\varphi}\right)^2 = \sigma \qquad (\sigma = \text{const})$$

$$\left(\frac{du}{d\varphi}\right)^2 + hu^2 = 2\,(E - \Pi) \qquad (E = \text{const}) \tag{4.12}$$

By replacing the variable $\varphi$ by $\tau$ $(d\tau = \sqrt{h}\,d\varphi)$ we can pass from Equations (4.4), (4.10) and (4.11) to Equations (3.9) to (3.11). We can utilize the symbolic formula for $\gamma$ and $u$

$$\frac{d^2}{d\varphi^2} = h\frac{d^2}{d\tau^2} - \frac{1}{u^2}\frac{\partial\Pi}{\partial\gamma}\frac{d\gamma}{d\tau}\frac{d}{d\tau} \tag{4.13}$$

If the independent variable $u$, is used, then we will find

$$\frac{dq}{du} = \frac{2}{u}(2\Pi - 2E + q) - 2\frac{\partial\Pi}{\partial u} \qquad \left(q = \left(\frac{du}{d\varphi}\right)^2\right)$$

$$q\left(\frac{d\gamma}{du}\right)^2 = -\sigma - \frac{1}{u^2}(1 - \gamma^2)(2\Pi - 2E + q) \qquad (E, \sigma = \text{const}) \tag{4.14}$$

The system of second order (4.14) is complete. It is suitable for studying the sections of the trajectories with monotonously varying radius $r$.

**5.. On the Lagrange's equations.** In the Lagrange's equations of second order ([11] p.283) the independent variable $t$ is assumed to be time. Let $q_s$ $(s = 1, \ldots, n)$ be the generalized coordinates, $T$ the kinetic energy

$$T = \frac{1}{2}\sum_{k,\,m=1}^{n} A_{km}(q_1, \ldots, q_n)\,\dot{q}_k\dot{q}_m \tag{5.1}$$

The dot denotes differentiation with respect to time. Let us introduce a new independent variable $\varphi$ in accordance with

$$d\varphi = \delta(q_1, \ldots, q_n)\,dt \tag{5.2}$$

Differentiation with respect to $\varphi$ in the present and the following Section will be denoted by a prime. Let

$$T_* = \frac{1}{2}\sum_{k,\,m=1}^{n} A_{km}(q_1, \ldots, q_n)\,q_k' q_m' \tag{5.3}$$

It can be proved that the equations of motion are of the form

$$\delta \frac{d}{d\varphi}\left(\delta \frac{\partial T_*}{\partial q_s'}\right) - \delta^2 \frac{\partial T_*}{\partial q_s} = Q_s \qquad (s = 1, \ldots, n) \tag{5.4}$$

Here $Q_s$ is the generalized force corresponding to the coordinate $q_s$.

**6. A different derivation of the equations of motion.** Equations (5.4) can be utilized for a shorter but less descriptive derivation of Equations (4.4), (4.10) and (4.11) and consequently (3.14). The kinetic energy of the point $M$ in the spherical system of coordinates is

$$T = {}^1/_2 \left[r^{\cdot 2} + r^2 \vartheta^{\cdot 2} + r^2 \sin^2 \vartheta \lambda^{\cdot 2}\right] \tag{6.1}$$

Assuming in (6.1)

$$r = u^{-1}, \qquad \cos\vartheta = \gamma \tag{6.2}$$

we find for $T_*$ (5.3)

$$T_* = \frac{1}{2}\left(\frac{u'^2}{u^4} + \frac{\gamma'^2}{u^2(1-\gamma^2)} + \frac{1-\gamma^2}{u^2}\lambda'^2\right) \tag{6.3}$$

Let us introduce the independent variable $\varphi$

$$d\varphi = u^2\, dt \tag{6.4}$$

Differentiation with respect to $\varphi$ will be denoted by a prime. Equations (5.4) then become

$$u^2 \frac{d}{d\varphi}\frac{u'}{u^2} + 2\frac{u'^2}{u} + \frac{u\gamma'^2}{1-\gamma^2} + u(1-\gamma^2)\lambda'^2 = -\frac{\partial \Pi}{\partial u} \tag{6.5}$$

$$u^2 \frac{d}{d\varphi}\frac{\gamma'}{1-\gamma^2} - \frac{\gamma'^2 \gamma u^2}{(1-\gamma^2)^2} + \gamma\lambda'^2 u^2 = -\frac{\partial \Pi}{\partial \gamma} \tag{6.6}$$

$$u^2 \frac{d}{d\varphi}(\lambda'(1-\gamma^2)) = 0 \tag{6.7}$$

Let us introduce an auxiliary quantity

$$h = \gamma'^2(1-\gamma^2)^{-1} + (1-\gamma^2)\lambda'^2 \tag{6.8}$$

After elementary transformations, Equation (6.5) becomes (4.4), while Equation (6.6) becomes (4.10). Differentiating $h$ (6.8) with respect to $\varphi$ there obtains Equation (4.11) in view of (6.7).

**7. Third form of the equations of motion.** In Equations (3.9), (3.10) and (3.12), the following substitution is made:

$$w = \frac{hu}{\mu} - 1, \qquad v = -\frac{h}{\mu}\frac{du}{d\tau} \qquad (\mu = \text{const}) \tag{7.1}$$

The equations for $w$ and $v$ are of the form

$$\frac{dw}{d\tau} = -v - \frac{2}{u\mu}\frac{\partial \Pi}{\partial \gamma}\frac{d\gamma}{d\tau}, \qquad \frac{dv}{d\tau} = w + \left(1 + \frac{1}{\mu}\frac{\partial \Pi}{\partial u}\right) + \frac{1}{u^2\mu}\frac{\partial \Pi}{\partial \gamma}\frac{d\gamma}{d\tau}\frac{du}{d\tau} \tag{7.2}$$

Assuming that the potential energy is

$$\Pi = -\mu u + \mu \Pi_*, \qquad |\Pi_*| \ll u \leqslant u_0 \tag{7.3}$$

and eliminating $u$ and $du/d\tau$ with the aid of (7.1), we obtain the equations of motion

$$\frac{dw}{d\tau} = -v - \frac{2p}{1+w}\frac{\partial \Pi_*}{\partial \gamma}\frac{d\gamma}{d\tau}, \qquad \frac{dv}{d\tau} = w + \frac{\partial \Pi_*}{\partial u} - \frac{vp}{(1+w)^2}\frac{\partial \Pi_*}{\partial \gamma}\frac{d\gamma}{d\tau} \tag{7.4}$$

$$\frac{dp}{d\tau} = -\frac{2p^2}{(1+w)^2}\frac{\partial \Pi_*}{\partial \gamma}\frac{d\gamma}{d\tau} \qquad \left(p \equiv \frac{h}{\mu}\right) \tag{7.5}$$

$$\frac{d^2\gamma}{d\tau^2}+\gamma=-\frac{p}{(1+w)^2}\left[1-\gamma^2-\left(\frac{d\gamma}{d\tau}\right)^2\right]\frac{\partial\Pi_*}{\partial\gamma} \tag{7.6}$$

We assume, that after differentiating $\Pi_*$, the variable $u$ is substituted by

$$u=(1+w)\,p^{-1} \tag{7.7}$$

Letting $\Pi_*\equiv 0$ in Equations (7.4) to (7.6), then, they have the obvious solution

$$p=\text{const},\qquad w=e\cos(\tau-\tau_1),$$
$$v=e\sin(\tau-\tau_1),\qquad \gamma=m\sin(\tau-\tau_2) \tag{7.8}$$

which corresponds to the Keplerian motion

$$r=u^{-1}=p\,(1+e\cos(\tau-\tau_1))^{-1} \tag{7.9}$$

The orbital inclination angle $i$ is found from the equality

$$\sin i=m \tag{7.10}$$

Therefore, the variable $p$ can be regarded as a focal parameter, while the quantity

$$e^2=w^2+v^2 \tag{7.11}$$

as the square of eccentricity of the osculating ellipse.

Equations (7.4) to (7.6) are convenient for investigating near circular orbits. They have two integrals which correspond to the integrals (3.13) and (3.15)

$$p\,[1-\gamma^2-(d\gamma/d\tau)^2]=c_1\qquad (c_1=\text{const}) \tag{7.12}$$

$$w^2+v^2=1-2p\,(c_2+\Pi_*)\qquad (c_2=\text{const}) \tag{7.13}$$

If $i$ is the orbital inclination angle of the osculating ellipse, then it follows from (3.8) that

$$\cos^2 i=(\mathbf{i}_3\cdot\mathbf{n})^2=\sin^2\chi\sin^2\vartheta=1-\gamma^2-(d\gamma/d\tau)^2 \tag{7.14}$$

The integrals (7.12) and (7.13) can be rewritten in the form

$$p\cos^2 i=c_1,\qquad 1-e^2=2p\,(c_2+\Pi_*) \tag{7.15}$$

If the focal parameter $p$ of the orbit is essentially constant, then the orbital inclination angle $i$ and the eccentricity $e$ of the osculating ellipse are nearly constant.

N o t e . From (7.15) follows the inequality

$$0\leqslant\cos^2 i=\frac{2c_1(c_2+\Pi_*)}{1-e^2}\leqslant 1 \tag{7.16}$$

If the quantity $\Pi_*$ during motion satisfies the condition

$$|\Pi_*-\Pi_0|\leqslant\delta,\qquad \Pi_0=\text{const} \tag{7.17}$$

then from the integrals (7.15) and (7.16) follow the bounds of the satellite trajectory with initial values of the parameters $i_0$, $e_0$, $r_0$

$$0<e_0<1,\qquad 0<i_0<{}^1\!/_2\pi,\qquad 0<r_0<\infty \tag{7.18}$$

During the entire motion the following inequalities are satisfied:

$$e^2\leqslant 1-(1-e_0^2)\cos^2 i_0\,(1-e-2r_0\delta)\,(1+e+2r_0\delta)^{-1} \tag{7.19}$$

$$1>\cos^2 i\geqslant\cos^2 i_0\,(1-e_0^2)\,(1-e-2r_0\delta)\,(1+e+2r_0\delta)^{-1} \tag{7.20}$$

$$r_0\frac{1-e}{1+(e+4r_0\delta)}\leqslant r\leqslant r_0\,\frac{1+e}{1-(e+4r_0\delta)} \tag{7.21}$$

Condition (7.16) must be fulfilled with (7.20) and (7.21). For the Earth the quantity $\delta$ is evaluated as

$$\delta \approx 1.3 \cdot 10^{-10} \, м^{-1} \tag{7.22}$$

During the motion, the satellite is within a certain toroidal body whose axis of symmetry coincides with the polar axis ($z$-axis).

The rough estimates (7.19) to (7.21) can be substantially improved for near-equatorial orbits.

**8. Pseudo-periodic trajectories of the satellite.** We will refer to a satellite trajectory as pseudo-periodic if it corresponds to a periodic solution of the system of equations (3.14) or (4.4), (4.10), (4.11), or (7.4) to (7.6). Let us consider a periodic solution of the system (3.14) with the period close to $2\pi$. We will investigate the segments of the corresponding pseudo-periodic trajectory enclosed by the sequential crossings of the equatorial plane from south to north. By assumption, the expression for the potential energy is independent of $\lambda$. Therefore, all segments of the trajectory are identical. They coincide exactly for a definite rotation about the $z$-axis. The pseudo-periodic trajectories are simplest in the motion of the satellite. We will find the necessary conditions for the existence of the pseudo-periodic trajectory. Let the potential energy in the gravitational field be of the form

$$\Pi(u, \gamma) = -\mu u - \frac{\varepsilon}{3} u^3 (1 - 3\gamma^2) - \frac{\varepsilon^2 \nu}{5} u^5 (3 - 30\gamma^2 + 35\gamma^4) + \ldots \tag{8.1}$$

Here $\varepsilon$ is a small parameter, and the quantities $\mu$, $\varepsilon$, $\nu$ are known (see, for example, [9 and 12], pp.75 and 77). In order to find the periodic solutions, we will utilize the system of equations (4.4), (4.10), (4.11) which for (8.1) becomes

$$d^2u / d\varphi + hu = \mu + \varepsilon u^2 (1 - 3\gamma^2) + \varepsilon^2 \nu u^4 (3 - 30\gamma^2 + 35\gamma^4) + \ldots$$

$$d^2\gamma / d\varphi^2 + h\gamma = (\gamma^2 - 1) [2\varepsilon u\gamma + \varepsilon^2 \nu u^3 (12\gamma - 28\gamma^3)] + \ldots \tag{8.2}$$

$$dh / d\varphi = -2 [2\varepsilon u\gamma + \varepsilon^2 \nu u^3 (12\gamma - 28\gamma^3)] \, d\gamma / d\varphi + \ldots$$

The instant of the satellite intersection of the equatorial plane will be taken as the initial time when $\gamma = 0$, $h = h_0$. We introduce new parameters and variables

$$\beta = \varepsilon\mu h_0^{-2}, \qquad \nu_1 = \mu\nu, \qquad z = uh_0\mu^{-1}$$

$$s = \varphi \, [h_0 (1 + \alpha_1\beta + \alpha_2\beta^2 + \ldots)]^{1/2}, \qquad q = hh_0^{-1} \tag{8.3}$$

We seek a periodic solution as a series in powers of $\beta$ [13]

$$z = z_0 + \beta z_1 + \beta^2 z_2 + \ldots, \qquad \gamma = \gamma_0 + \beta\gamma_1 + \beta^2\gamma_2 + \ldots$$

$$q = q_0 + \beta q_1 + \beta^2 q_2 + \ldots \tag{8.4}$$

For the terms in (8.4) we find Equations

$$\frac{d^2 z_0}{ds^2} + z_0 = 1, \qquad \frac{d^2\gamma_0}{ds^2} + \gamma_0 = 0, \qquad \frac{dq_0}{ds} = 0 \tag{8.5}$$

The first approximation equations become

$$\frac{d^2z_1}{ds^2} + z_1 + \alpha_1 \frac{d^2z_0}{ds^2} + q_1 z_0 = z_0^2(1 - 3\gamma_0^2)$$

$$\frac{d^2\gamma_1}{ds^2} + \gamma_1 + \alpha_1 \frac{d^2\gamma_0}{ds^2} + q_1\gamma_0 = 2z_0(\gamma_0^3 - \gamma_0) \tag{8.6}$$

$$\frac{dq_1}{ds} = -4z_0\gamma_0 \frac{d\gamma_0}{ds}$$

The second approximation equations are

$$\frac{d^2z_2}{ds^2} + z_2 = -\alpha_1 \frac{d^2z_1}{ds^2} - \alpha_2 \frac{d^2z_0}{ds^2} - q_1 z_1 - q_2 z_0 - 6z_0^2\gamma_0\gamma_1 +$$

$$+ 2z_0 z_1(1 - 3\gamma_0^2) + \nu_1 z_0^4(3 - 30\gamma_0^2 + 35\gamma_0^4)$$

$$\frac{d^2\gamma_2}{ds^2} + \gamma_2 = -\alpha_1 \frac{d^2\gamma_1}{ds^2} - \alpha_2 \frac{d^2\gamma_0}{ds^2} - q_1\gamma_1 - q_2\gamma_0 + \tag{8.7}$$

$$+ 2z_0\gamma_1(3\gamma_0^2 - 1) + 2z_1(\gamma_0^3 - \gamma_0) + \nu_1 z_0^3(-28\gamma_0^5 + 40\gamma_0^4 - 12\gamma_0)$$

$$\frac{dq_2}{ds} = -4z_0\gamma_0\frac{d\gamma_1}{ds} - 4z_0\gamma_1\frac{d\gamma_0}{ds} - 4z_1\gamma_0\frac{d\gamma_0}{ds} -$$

$$-2\nu_1 z_0^3(12\gamma_0 - 28\gamma_0^3)\frac{d\gamma_0}{ds}$$

The generating solution will be assumed

$$z_0 = 1 + e\cos(s - s_0), \qquad \gamma_0 = m\sin s, \qquad q_0 = 1 \tag{8.8}$$

From the last equation in (8.6), we find

$$q_1 = m^2\,[\cos 2s + {}^1/_3 e\cos(3s - s_0) + e\cos(s + s_0)] \tag{8.9}$$

The conditions of periodicicty for $z_1$, $\gamma_1$ lead to the equalities

$$e(\alpha_1 + 2 - 3m^2) = 0, \qquad m(\alpha_1 + 2m^2 - 2) = 0 \tag{8.10}$$

They can be satisfied in three cases.

1. For $e = 0$. The generating orbit is circular.

2. For $m = 0$. The generating orbit lies in the equatorial plane. In this case, Equations (8.2) can be integrated .

3. For $5m^2 - 4 = 0$. The generating orbit has an angle of inclination

$$i = \cos^{-1}(0.2\sqrt{5}) \approx 63°28'$$

In satisfying (8.10), we find $z_1$ and $\gamma_1$

$$z_1 = (1 + {}^1/_2 e^2)(1 - {}^3/_2 m^2) - {}^1/_8 m^2e^2\cos 2s_0 - {}^1/_{36} m^2(6 + e^2)\cos 2s -$$
$$- {}^1/_{12} e^2(2 - 3m^2)\cos(2s - 2s_0) - {}^1/_{12} m^2 e\cos(3s - s_0) -$$
$$- {}^1/_{72} m^2e^2\cos(4s - 2s_0) \tag{8.11}$$

$$\gamma_1 = {}^1/_4 me(5m^2 - 4)\sin s_0 - {}^1/_{36} me(11m^2 - 12)\sin(2s - s_0) +$$
$$+ {}^1/_4 m^3 e\sin(2s + s_0) + {}^1_8/m^3\sin 3s + {}^1/_{36} m^3 e\sin(4s - s_0) \tag{8.12}$$

Following Sections consider the conditions for which there exists a periodic solution of the system (8.7).

**9. The generating orbit with an angle of inclination of 63°28'.** Assuming that $0 < e < 1$ and $5m^2 = 4$ we find $q_2$

$$q_2 = {}^2/_{75}e^2 (5 - 18\nu_1)\, s \sin 2s_0 + \ldots \tag{9.1}$$

Dots indicate periodic terms. From data in [12] (p.77) we find the value of $\nu_1 = 0.562$. It follows from (9.1) that the generating elliptic orbit for a pseudo-periodic trajectory must have the perigee and apogee located either in the equatorial plane or at the most northern or southern points. At the same time

$$\sin 2s_0 = 0 \tag{9.2}$$

Assuming that in (8.8) $s_0 = 0$, we find the periodicity conditions for $z_2, \gamma_2$ as

$$e\,[\alpha_2 - {}^8/_{25} - {}^{173}/_{450}e^2 + \nu_1 ({}^2/_{25} - {}^{13}/_5 e^2)] = 0 \tag{9.3}$$

$$\alpha_2 - {}^{14}/_{75} + {}^{67}/_{450}e^2 + \nu_1 ({}^{24}/_{25} - {}^3/_{25}\, e^2) = 0 \tag{9.4}$$

Conditions (9.3) and (9.4) cannot be fulfilled if $e \neq 0$. After eliminating $\alpha_2$ and substituting $\nu_1$, there results the impracticable condition

$$0.38 + 1.83e^2 = 0 \tag{9.5}$$

Assuming that in (8.8) $s_0 = 0.5\pi$, we find the periodicity conditions for $z_2, \gamma_2$ as

$$e\,[-\alpha_2 + {}^4/_{75} - {}^{11}/_{90}e^2 + \nu_1 ({}^{72}/_{25} + 3e^2)] = 0 \tag{9.6}$$

$$-\alpha_2 + {}^{14}/_{75} + {}^{31}/_{90}e^2 + \nu_1 (-{}^{24}/_{25} - 3e^2) = 0 \tag{9.7}$$

After eliminating $\alpha_2$ and substituting $\nu_1$, there results the impracticable condition

$$2.03 + 2.91e^2 = 0 \tag{9.8}$$

The cases for $s_0 = \pi$, $s_0 = 1.5\pi$ are obtained from the considered cases by replacement of $e$ by $-e$.

The final result shows that the pseudo-periodic trajectories cannot exist if the generating elliptic orbit has an inclination angle $\iota = 63°28'$ and eccentricity $e > 0$.

**10. The case of a circular generating orbit.** It will be shown that in this case, there exists an entire family of pseudo-periodic trajectories of a satellite, i.e. a family of periodic solutions for the system (8.2). Let us assume that the potential energy is of the form

$$\Pi(u, \gamma) = -\mu u - \varepsilon u^3 \Pi_1 (u^2, \gamma^2, \varepsilon) \tag{10.1}$$

where $\varepsilon$ is a small parameter $\Pi_1(u^2, \gamma^2, \varepsilon)$ is a power series of all variables which is convergent for

$$|u| < u_{10}, \qquad |\gamma| \leqslant 1, \qquad |\varepsilon| \leqslant \varepsilon_1 \tag{10.2}$$

Equations (4.4), (4.10) and (4.11) become

$$\frac{d^2u}{d\varphi_2} + hu = \mu + 3\varepsilon u^2 \Pi_1 (u^2, \gamma^2, \varepsilon) + 2\varepsilon u^4 \frac{\partial \Pi_1 (u^2, \gamma^2, \varepsilon)}{\partial u^2} \qquad (h > 0)$$

$$\frac{d^2\gamma}{d\varphi^2} + h\gamma = 2\varepsilon (1 - \gamma^2)\, u\gamma \frac{\partial \Pi_1 (u^2, \gamma^2, \varepsilon)}{d\gamma^2}, \qquad \frac{dh}{d\varphi} = 2\varepsilon u \frac{\partial \Pi_1 (u^2, \gamma^2, \varepsilon)}{d\gamma^2} \frac{d\gamma^2}{d\varphi} \tag{10.3}$$

Let for $\varphi = \varphi_0$ we have $\gamma = 0$, $h = h_0$. Performing the change of the independent variable

$$s = [h_0(1 + \alpha_1\varepsilon + \alpha_2\varepsilon^2 + \ldots)]^{1/2}(\varphi - \varphi_0) \tag{10.4}$$

We seek a periodic solution of system (10.3) in the form of series

$$u = u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \ldots, \quad \gamma = \gamma_0 + \varepsilon\gamma_1 + \varepsilon^2\gamma_2 + \ldots,$$
$$h = h_0(q_0 + \varepsilon q_1 + \varepsilon^2 q_2 + \ldots) \tag{10.5}$$

From (10.3), we find the generating solution which corresponds to the circular orbit of the satellite

$$u_0 = \mu h_0^{-1}, \qquad \gamma_0 = m \sin s \qquad (m = \text{const}, \ m \neq 0), \qquad q_0 = 1 \tag{10.6}$$

Let us introduce the classes of trigonometric polynomials (convergent series) of the type

$$C_1 = \sum_n a_{1n} \cos(2n+1)s, \qquad C_2 = \sum_n a_{2n} \cos 2ns$$
$$S_1 = \sum_n b_{1n} \sin(2n+1)s, \qquad S_2 = \sum_n b_{2n} \sin 2ns \tag{10.7}$$

The classes of functions (10.7) form a commutative subgroup with a multiplication law for which the product given by the table

| | $C_1$ | $C_2$ | $S_1$ | $S_2$ |
|---|---|---|---|---|
| $C_1$ | $C_2$ | $C_1$ | $S_2$ | $S_1$ |
| $C_2$ | $C_1$ | $C_2$ | $S_1$ | $S_2$ |
| $S_1$ | $S_2$ | $S_1$ | $C_2$ | $C_1$ |
| $S_2$ | $S_1$ | $S_2$ | $C_1$ | $C_2$ |

(10.8)

yields equalities, for example $C_1 S_1 = S_2$; these equalities are regarded in the following sense. For any trigonometric polynomials $\chi_1(s), \chi_2(s)$ such that $\chi_1 \in C_1, \chi_2 \in S_1$, their product $\chi_1\chi_2 \in S_2$. Analogous interpretation is given by the symbolic equalities

$$\frac{dC_1}{ds} = S_1, \qquad \frac{dC_2}{ds} = S_2, \qquad \int S_2 ds = C_2, \qquad \frac{d^2C_2}{ds^2} = C_2, \ldots$$

We introduce the functions $\Phi_j(u, \gamma, \varepsilon)$

$$\Phi_1(u, \gamma, \varepsilon) \equiv \left[\mu + 3\varepsilon u \Pi_1(u^2, \gamma^2, \varepsilon) + 2\varepsilon u^4 \frac{\partial \Pi_1(u^2, \gamma^2, \varepsilon)}{\partial u^2}\right]\frac{1}{h_0}$$
$$\Phi_2(u, \gamma, \varepsilon) \equiv 2\varepsilon(1 - \gamma^2) u\gamma \frac{\partial \Pi_1(u^2, \gamma^2, \varepsilon)}{\partial \gamma^2}\frac{1}{h_0} \tag{10.9}$$
$$\Phi_3(u, \gamma, \varepsilon) \equiv 2\varepsilon u \frac{\partial \Pi_1(u^2, \gamma^2, \varepsilon)}{\partial \gamma^2}\frac{\partial \gamma^2}{\partial s}\frac{1}{h_0}$$

Let us prove now that the functions $u_j, \gamma_j, q_j$ can always be selected by the periodic functions $s$ such that

$$u_j \in C_2, \qquad \gamma_j \in S_1, \qquad q_j \in C_2 \qquad (j = 0, 1, 2, \ldots) \tag{10.10}$$

Equation (10.10) follows from (10.6) when $j = 0$.

Let (10.10) be fulfilled for $(j = 0, 1, \ldots, n-1)$. Introduce the notation

$$u_n^* = \sum_{j=0}^{n-1} \varepsilon^j u_j, \qquad \gamma_n^* = \sum_{j=0}^{n-1} \varepsilon^j \gamma_j, \qquad q_n^* = \sum_{j=0}^{n-1} \varepsilon^j q_j \tag{10.11}$$

From (10.8) to (10.10) follow the relationships

$$\Phi_1(u_n^*, \gamma_n^*, \varepsilon) \in C_2, \qquad \Phi_2(u_n^*, \gamma_n^*, \varepsilon) \in S_1, \qquad \Phi_3(u_n^*, \gamma_n^*, \varepsilon) \in S_2 \tag{10.12}$$

Differentiation of $\Phi_i(y_n^*, \gamma_n^*, \varepsilon)$ with respect to $\varepsilon$ does not alter the class of the function.

Differential equations for $u_n, \gamma_n, q_n$ are

$$\frac{d^2u_n}{ds^2} + u_n = -\sum_{j=0}^{n-1}\left(\alpha_{n-j}\frac{d^2u_j}{ds^2} + q_{n-j}u_j\right) + \frac{1}{n!}\left.\frac{\partial^n\Phi_1(u_n^*, \gamma_n^*, \varepsilon)}{\partial\varepsilon^n}\right|_{\varepsilon=0} \quad (10.13)$$

$$\frac{d^2\gamma_n}{ds^2} + \gamma_n = -\sum_{j=1}^{n-1}\alpha_{n-j}\frac{d^2\gamma_j}{ds^2} - \sum_{j=0}^{n-1} q_{n-j}\gamma_{j-1} - \alpha_n m\sin s + \frac{1}{n!}\left.\frac{\partial^n\Phi_2(u_n^*, \gamma_n^*, \varepsilon)}{\partial\varepsilon^n}\right|_{\varepsilon=0}$$

$$\frac{dq_n}{ds} = \frac{1}{n!}\left.\frac{\partial^n\Phi_3(u_n^*, \gamma_n^*, \varepsilon)}{\partial\varepsilon^n}\right|_{\varepsilon=0} \quad (10.14)$$

It follows from (10.12) that the right-hand side in Equation (10.13) belongs to the $S_2$ class, and consequently $q_n \in C_2$. The right-hand side of (10.14) belongs to the $S_1$ class. Considering the constants $\alpha_1, \ldots, \alpha_{n-1}$ known, we select $\alpha_n$ such that the right-hand side does not contain the term with $\sin s$ in its Fourier expansion. For $m \neq 0$ ($m = \sin i$) the constant $\alpha_n$ is selected uniquely. The function $\gamma_n$ can always be selected belonging to class $S_1$. The selection will be unique if $\gamma_n$ does not contain $\sin s$ in its expansion. The right-hand side of Equation (10.13) belongs to the class $C_2$, and therefore, the function $u_n$ can always be uniquely selected such that $u_n \in C_2$. The validity of the relationships (10.10) is proved for all $n$. If an auxiliary condition is applied

$$q_n = 0, \qquad s = 0, \qquad (n = 1, 2, \ldots) \quad (10.15)$$

then the selection of the functions $u_n, \gamma_n, q_n$ will be unique.

With the above indicated selection of the functions, we can prove the convergence of the series (10.4), (10.5) if $h_0$ is sufficiently large and $\varepsilon > 0$ is sufficiently small. In investigating the convergence it is convenient to utilize the normalized space $C[0, 2\pi]$ of the functions of $s$ continuous on the segment $[0, 2\pi]$ with the norm

$$\|u\| = \max |u(s)|, \qquad s \in [0, 2\pi] \quad (10.16)$$

The proof of convergence is involved and is not presented here. From the proof follows the existence of a family of periodic solutions for the system of equations (10.3) which depends on three arbitrary parameters $h_0, m, \varphi_0$.

Through each point of the space around the Earth passes one parametric family of near circular pseudo-periodic trajectories.

**11. The influence of the Earth's pyriform shape. The generating orbit with the inclination angle of 63°28'.** The study of satellite motions proved the existence of a third harmonic related to the pyriform shape of the Earth [12] (p.75). Let the potential energy expression, different from (8.1), be of the form

$$\Pi(u, \gamma) = -\mu u - {}^1/_3\,\varepsilon u^3(1 - 3\gamma^2) - \varepsilon^2\xi u^4(3\gamma - 5\gamma^3) - $$
$$- {}^1/_5\,\varepsilon^2\nu u^5(3 - 30\gamma^2 + 35\gamma^4) + \ldots \quad (11.1)$$

Repeating the arguments of the Sections 8 and 9, we find the condition $e\xi = 0$, auxiliary to the conditions (9.3) and (9.4), from the periodicity conditions for $z_2, \gamma_2$ in the generating solution (8.8) with $m = 0.4\sqrt{5}$, $s_0 = 0$.

The case $s_0 = 0.5\pi$ is more interesting. Conditions (9.6) and (9.7) alter in form. After eliminating $\alpha_2$, there results the equality

$$ {}^2/_{15} + {}^{21}/_{45}\,e^2 + 3me\xi h_0 + \nu_1[-{}^{96}/_{25} - 6e^2] = 0 \quad (11.2)$$

Fulfillment of (11.2) is necessary for the existence of pseudo-periodic trajectories. From the data in [12] (p.79) the values of the constants are

$$\nu_1 = 0.40 \pm 0.02, \qquad \xi h_0 = (0.45 \pm 0.05)\, p_0 R^{-1} \tag{11.3}$$

Here $p_0$ is the focal parameter of the generating orbit, $R$ is the radius of the Earth. The orbit perigee must lie outside the Earth, and therefore, the following inequality must be fulfilled

$$p_0 > R\,(1 + e) \tag{11.4}$$

The equality (11.2) becomes

$$e^2 - 0.63\, e\eta + 0.73 = 0, \qquad \eta \equiv p_0 R^{-1} > 0 \tag{11.5}$$

The condition (11.4) becomes

$$e^2 + 0.73 > (1 + e)\, 0.63\, e \tag{11.6}$$

and is always fulfilled. Condition (11.2) which is impracticable for $\xi = 0$, can be satisfied by the choice of $h_0$ or the selection of $\eta$ in (11.6). Since $e > 0$, $s_0 = 0.5\pi$ the generating orbit corresponding to (8.8) is elongated to the south. Its pedigree is at the northermost point and apogee at the southernmost point. The increase of the orbit size increases the relative influence of the third harmonic in (11.1) compared to the fourth harmonic. The smallest value of the focal parameter $p_0 \approx 2.7R$ is obtained for $e \approx 0.85$.

B a s i c  c o n c l u s i o n . As a consequence of the Earth pyriform shape, there can exist pseudo-periodic trajectories (with accuracy uo to $\varepsilon^3$), for which the generating elliptic orbit has the angle of inclination $\iota \approx 63°28'$ and which is elongated southward with perigee at the northernmost point. The eccentricity $e$ and the focal parameter $p_0$ must be related by the approximate equality (11.5). The smaller the eccentricity $e$, the larger is the focal parameter $p_0$.

**12. The influence of the Earth's pyriform shape on the near circular pseudo-periodic trajectories.** The proof of the possibility of constructing pseudo-periodic trajectories utilized in Section 10 is not usable for the potential energy $\Pi(u, \gamma)$ of the form (11.1). The presence of the third harmonic imposes auxiliary conditions on the parameters of the generating solution (10.6).

From the periodicity condition for $u_2$ there follows the equality

$$4 - 5m^2 = 0 \tag{12.1}$$

The inclination angle of the generating orbit must be approximately equal to 63°28'.

P r a c t i c a l  c o n c l u s i o n . The noncentral nature of the Earth's gravitational field has the smallest influence upon near circular satellite trajectories with the inclination angle of 63°28'.

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